

## Discrete spectral shift in an anisotropic universe

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### Abstract

In this paper we calculate the particle creation as seen by a stationary observer in an anisotropic universe. By using an observer and geometry dependent time to quantise a massive scalar field we show that a discrete energy spectrum shift occurs. The length scale associated with the geometry provides the energy scale by which the spectrum is shifted. The  $\beta(p, q)$  coefficient for the Bogolubov transformation calculated is proportional to a series of delta functions whose argument contains  $p$  and  $q$  and half multiples of the root of the curvature.

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## I. INTRODUCTION

The problems surrounding the construction of quantum fields in curved spacetime backgrounds have seen renewed interest [1]. In particular much of this interest is centered around attempts to define particles in curved backgrounds and also various means of normal ordering expectation values to obtain physically reasonable finite results. This is in contrast to much of the earlier work which involved calculations of detector response functions or elaborate means of regularizing and renormalizing expectation values of the stress tensor. Of course these two problems are intimately related, if there are regions where one can construct a Fock space, and thus have a particle interpretation, a natural normal ordering or vacuum subtraction procedure exists. The difference in the recent literature is that the regions where one expects to be able to do this are no longer required to be flat.

The approach we use in this paper is that of Capri and Roy [2]. This is a coordinate independent geometrical approach where the geometry determines a preferred time with which to quantize the field. This preferred direction of time is along a normal to a spacelike surface consisting of those spacelike geodesics which are orthogonal to the observers 4-velocity. In this way the construction depends only on the geometry, the observers position, and the tangent to the observer's worldline. This approach is very similar to that of [3] where they also construct a spacelike surface which is geodesic. The major difference is that here a local Poincaré invariance is also used.

## II. THE MODEL

The model we investigate in this paper is an anisotropic  $3 + 1$  generalization of  $1 + 1$  de Sitter space of constant curvature. The simplest generalization being just the addition of a 2-plane. Specifically we are investigating particle creation due to the gravitational field which is described by the metric,

$$ds^2 = dT^2 - e^{\lambda T} (dX^1)^2 - (dX^2)^2 - (dX^3)^2. \quad (2.1)$$

More precisely we investigate the particle creation as observed by an observer stationary with respect to the coordinates  $(T, X^1, X^2, X^3)$

To follow the prescription of [2] we first find the geodesics in this spacetime. The first integrals of the geodesics are:

$$\frac{dX^1}{ds} = \frac{c_1}{e^{\lambda T}}, \quad \frac{dX^2}{ds} = c_2, \quad \frac{dX^3}{ds} = c_3, \quad \frac{dT}{ds} = \sqrt{\epsilon + \frac{c_1^2}{e^{\lambda T}} + c_2^2 + c_3^2} \quad (2.2)$$

where  $\epsilon = \pm 1$  depending on whether the geodesic is timelike or spacelike respectively.

The preferred coordinates on the surface are constructed using a 4-bein of orthogonal basis vectors at  $P_0$ , the observers position. We choose these vectors to be,

$$e_0(P_0) = (1, 0, 0, 0) \quad e_1(P_0) = (0, e^{-\lambda \frac{T_0}{2}}, 0, 0) \quad e_2(P_0) = (0, 0, 1, 0) \quad e_3(P_0) = (0, 0, 0, 1). \quad (2.3)$$

In this way the tangent to the chosen observer's worldline at  $P_0$  corresponds to  $e_0(P_0)$ .

To construct the spacelike surface orthogonal to the tangent of the observer's worldline we therefore require that,

$$\left. \frac{dT}{ds} \right|_{P_0} = 0 \quad \text{which implies} \quad \frac{c_1^2}{e^{\lambda T_0}} + c_2^2 + c_3^2 = 1 \quad (2.4)$$

The preferred coordinates on the spacelike hypersurface are chosen to be Riemann coordinates based on the observer's position  $P_0 = (T_0, X_0^1, X_0^2, X_0^3)$ . With  $p^\mu$  given by the tangent vector, at  $P_0$ , to the geodesic connecting  $P_0$  to  $P_1$ . The point  $P_1$  is the point at which the timelike geodesic "dropped" from an arbitrary point  $P = (T, X^1, X^2, X^3)$  intersects the spacelike surface orthogonally. The Riemann coordinates  $\eta^\alpha$  of the point  $P_1$  are given by,

$$sp^\mu = \eta^\alpha e_\alpha^\mu(P_0) \quad (2.5)$$

where  $s$  is the distance along the geodesic  $P_0 - P_1$ . Using  $e_\alpha^\mu e_{\beta\mu} = \eta_{\alpha\beta}$  (Minkowski metric), and the orthogonality of  $p^\mu$  to  $e_0(P_0)$  we have

$$\eta^0 = sp^\mu e_\mu^0(P_0) \quad \eta^i = -sp^\mu e_\mu^i(P_0). \quad (2.6)$$

The surface  $S_0$  is just the surface  $\eta^0 = 0$  and the coordinates  $x^i$  are

$$x^1 = s \frac{c_1}{\sqrt{e^{\lambda T_0}}} \quad x^2 = s c_2 \quad x^3 = s c_3 \quad (2.7)$$

where  $s$  is the geodesic distance between the points  $P_0$  and  $P_1$ .

The direction of time is given by the normal to the spacelike hypersurface and the preferred time  $t$  for the arbitrary point  $P$  is given by the proper distance along this timelike geodesic connecting  $P$  to  $P_1$ . The timelike geodesic is also determined by (2.2) except with  $\epsilon = -1$  and a different choice of the constants which we denote by  $b_i$ . The condition that the geodesic connecting  $P$  to  $P_1$  is normal to the spacelike hypersurface requires that

$$\sqrt{\left(1 + \frac{b_1^2}{e^{\lambda T_1}} + b_2^2 + b_3^2\right)} \sqrt{\left(\frac{c_1^2}{e^{\lambda T_1}} - \frac{c_1^2}{e^{\lambda T_0}}\right)} = \frac{b_1 c_1}{e^{\lambda T_1}} + b_2 c_2 + b_3 c_3. \quad (2.8)$$

We can now calculate the dependence of  $(T, X^1, X^2, X^3)$  on the preferred coordinates  $(t, x^1, x^2, x^3)$  and then calculate the metric in its preferred form. To calculate this dependence we must use the above equations for  $x^i$  (2.7) and also calculate the change in the coordinates  $X^i$  along the spacelike and timelike geodesics which ultimately connect  $P_0$  to  $P$ .

$$\begin{aligned} X^1 &= X_0^1 + \int_{T_0}^{T_1} dT \frac{c_1}{e^{\lambda T_1}} \left( \frac{c_1^2}{e^{\lambda T}} - \frac{c_1^2}{e^{\lambda T_0}} \right)^{-\frac{1}{2}} + \int_{T_1}^T dT' \frac{b_1}{e^{\lambda T'}} \left( 1 + \frac{b_1^2}{e^{\lambda T'}} + b_2^2 + b_3^2 \right)^{-\frac{1}{2}} \\ X^2 &= X_0^2 + \int_{T_0}^{T_1} dT \frac{c_2}{e^{\lambda T_1}} \left( \frac{c_1^2}{e^{\lambda T}} - \frac{c_1^2}{e^{\lambda T_0}} \right)^{-\frac{1}{2}} + \int_{T_1}^T dT' \frac{b_2}{e^{\lambda T'}} \left( 1 + \frac{b_1^2}{e^{\lambda T'}} + b_2^2 + b_3^2 \right)^{-\frac{1}{2}} \\ X^3 &= X_0^3 + \int_{T_0}^{T_1} dT \frac{c_3}{e^{\lambda T_1}} \left( \frac{c_1^2}{e^{\lambda T}} - \frac{c_1^2}{e^{\lambda T_0}} \right)^{-\frac{1}{2}} + \int_{T_1}^T dT' \frac{b_3}{e^{\lambda T'}} \left( 1 + \frac{b_1^2}{e^{\lambda T'}} + b_2^2 + b_3^2 \right)^{-\frac{1}{2}} \end{aligned} \quad (2.9)$$

and  $t$  the proper distance along  $P - P_1$

$$t = \int_{T_1}^T dT' \left( 1 + \frac{b_1^2}{e^{\lambda T'}} + b_2^2 + b_3^2 \right)^{-\frac{1}{2}}. \quad (2.10)$$

At this point we can see that if we choose  $b_2 = b_3 = 0$  this just corresponds to aligning the spacelike hypersurfaces so that  $X^2 = X_1^2$  and  $X^3 = X_1^3$ . This simplifies the analysis considerably and gives the expected result that,

$$X^2 = X_0^2 + x^2 \quad \text{and} \quad X^3 = X_0^3 + x^3. \quad (2.11)$$

The only non-trivial part of the transformation therefore involves  $(T, X^1)$  and  $(t, x^1)$ . By performing the above integral for  $X^1$  and inverting the  $t$  integral one is left with the coordinate transformations

$$\begin{aligned} e^{\frac{\lambda}{2}(T-T_0)} &= \sinh\left(\frac{\lambda t}{2}\right) + \cosh\left(\frac{\lambda t}{2}\right) \cos\left(\frac{\lambda x^1}{2}\right) \\ \frac{\lambda}{2}(X^1 - X_0^1)e^{\lambda \frac{T}{2}} &= -\cosh\left(\frac{\lambda t}{2}\right) \sin\left(\frac{\lambda x^1}{2}\right). \end{aligned} \quad (2.12)$$

In terms of the preferred coordinates  $(t, x^i)$  the metric now has the form,

$$ds^2 = dt^2 - \cosh^2\left(\frac{\lambda t}{2}\right)(dx^1)^2 - (dx^2)^2 - (dx^3)^2. \quad (2.13)$$

This result is of course not a surprise to anyone familiar with the different forms of de Sitter space in  $1 + 1$  dimensions. Unfortunately the usual analysis does not deal with the observer dependent nature of the coordinate transformations. We will see that this is in fact where the interesting physics comes from. Indeed if one proceeds to quantize the field on  $t = \text{constant}$  surfaces it is easy to see that all these surfaces can be made to look like Minkowski space. The point is that they cannot be made to all look like Minkowski space simultaneously. It would therefore seem obvious that the physics is going to be determined not by the form of the metric on a particular surface but by the transformations relating one surface's preferred coordinates to another surface's preferred coordinates.

### III. MODES AND QUANTIZATION

In the coordinates constructed in the last section the non-minimally coupled massive Klein Gordon equation is

$$\partial_t^2 \phi + \frac{1}{\sqrt{g}} \partial_t (\sqrt{g}) \partial_t \phi + \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij}) \partial_j \phi + (m^2 + \xi R) \phi = 0 \quad (3.1)$$

This equation is strictly hyperbolic so long as  $g^{ij}$  does not change sign. The solutions are therefore uniquely determined by the initial data.

To quantize a scalar field on the  $t = 0$  surface we now define the positive frequency modes in the neighbourhood of this surface. The positive frequency modes are defined as those which satisfy the initial conditions,

$$\phi_k^+(t, \mathbf{x})|_{t=0} = A_k(0, \mathbf{x}) \quad \text{and} \quad \partial_t(\phi_k^+(t, \mathbf{x}))|_{t=0} = -i\omega_k(0)A_k(0, \mathbf{x}) \quad (3.2)$$

Where  $A_k(t, \mathbf{x})$  are the instantaneous eigenmodes of the spatial part of the Laplace-Beltrami operator, and  $\omega_k(t)^2$  are the corresponding eigenvalues.

$$\left[ \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) + m^2 + \xi R \right] A_k(t, \mathbf{x}) = \omega_k^2(t) A_k(t, \mathbf{x}). \quad (3.3)$$

Henceforth we just write  $\omega_k$  for  $\omega_k(0)$ . Due to the simple form of  $g_{\mu\nu}$  at  $t = 0$  the eigenmodes and values take on the simple form,

$$A_k(0, \mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} \quad (3.4)$$

$$\omega_k^2(0) = \mathbf{k}^2 + m^2 + \xi R.$$

Near the surface  $t = 0$  the second term of (3.1) vanishes to  $O(t^2)$ , this implies that the initial conditions for the time dependence of the field which are also good to  $O(t^2)$ .

To impose these initial conditions we must find a complete set of modes for the entire wave operator. Because the differential equation is separable we look for solutions of the form  $f_k(t)e^{i\mathbf{k} \cdot \mathbf{x}}$ . The differential equation satisfied by the  $f_k(t)$  is then,

$$\partial_t^2 f_k(t) + \frac{\lambda}{2} \tanh\left(\frac{\lambda t}{2}\right) \partial_t f_k(t) + \left( k_1^2 \text{sech}^2\left(\frac{\lambda t}{2}\right) + k_2^2 + k_3^2 + m^2 + \xi R \right) f_k(t) = 0 \quad (3.5)$$

The positive frequency modes are those whose “time” part satisfy the above differential equation and the initial conditions

$$f_k(0) = 1 \quad \text{and} \quad \dot{f}_k(0) = -i\omega_k. \quad (3.6)$$

The positive frequency modes are given in terms of hypergeometric functions  $H(a, b, c, x)$  by

$$\phi_k^+(t, \mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} \text{sech}\left(\frac{\lambda t}{2}\right)^{2n} \left\{ H\left(\alpha, \beta, \frac{1}{2}, \tanh^2\left(\frac{\lambda t}{2}\right)\right) - i \frac{2\omega_k}{\lambda} \tanh\left(\frac{\lambda t}{2}\right) H\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}, \frac{3}{2}, \tanh^2\left(\frac{\lambda t}{2}\right)\right) \right\} \quad (3.7)$$

where

$$\begin{aligned}
n &= \frac{1}{4} - \frac{i}{\lambda} \sqrt{k_2^2 + k_3^2 + m^2 + \xi R - \frac{\lambda^2}{16}} \\
\alpha &= -\frac{k_1}{\lambda} + \frac{1}{4} - \frac{i}{\lambda} \sqrt{k_2^2 + k_3^2 + m^2 + \xi R - \frac{\lambda^2}{16}} \\
\beta &= \frac{k_1}{\lambda} + \frac{1}{4} - \frac{i}{\lambda} \sqrt{k_2^2 + k_3^2 + m^2 + \xi R - \frac{\lambda^2}{16}}.
\end{aligned} \tag{3.8}$$

We can now write out the field which has been quantized on surface 1 as,

$$\Psi_1 = \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{2\omega_k}} \left\{ \phi_k^+(t, \mathbf{x}) a_1(k) + \phi_k^{+*}(t, \mathbf{x}) a_1^\dagger(k) \right\} \tag{3.9}$$

#### IV. PARTICLE CREATION

To investigate particle creation in the model universe as observed by an observer stationary with respect to the original coordinates  $(T, X^1, X^2, X^3)$  we calculate the Bogolubov transformation relating the annihilation and creation operators from two different surfaces of quantization that the observer passes through. To calculate the coefficients of this transformation we equate the same field from two different quantizations on a common surface,

$$\Psi_1(t, x) = \Psi_2(t'(t, x), x'(t, x)). \tag{4.1}$$

Here  $\Psi_1(t, x)$  is the field written out explicitly in (3.9) and  $\Psi_2(t', x')$  is the same field which has been quantized on a second surface  $t' = 0$ . The “second” field is therefore quantized for the same observer as the first but at some later time  $T'_0$ . At this time the remark made at the end of the second section becomes clearer. All the physics of the observations made by this observer are determined by the functions  $t'(t, x)$ ,  $x'(t, x)$  and the derivatives of these functions with respect to  $t$ . In this way the geometry of the spacetime via the coordinate independent prescription we have used, determines the spectrum of created particles.

For simplicity we calculate the Bogolubov transformation by “matching” the field and its first derivative with respect to  $t$  at  $t = 0$ .

$$\begin{aligned}
a_1(k) &= \frac{i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \{ -i\omega_k \Psi_1(0, x) + (\partial_t \Psi_1(t, x))|_{t=0} \} \\
&= \frac{i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \{ -i\omega_k \Psi_2(t'(0, x), x'(0, x)) + (\partial_t \Psi_2(t'(t, x), x'(t, x)))|_{t=0} \}
\end{aligned} \tag{4.2}$$

Using this equation, we can write out the Bogolubov transformation in the form

$$a_1(k) = \int d^3p \alpha(k, p) a_2(p) + \int d^3p \beta(k, p) a_2^\dagger(p). \tag{4.3}$$

The spectrum of created particles is determined by  $|\beta(k, p)|^2$ . Writing out  $\beta(k, p)$  explicitly we find it has some interesting properties due to it's dependence on the inverse relations  $t'(t, x), x'(t, x)$ ,

$$\beta(k, p) = \frac{-i}{2\pi} \delta(p_2 + k_2) \delta(p_3 + k_3) \int dx_1 \frac{e^{ik_1 x_1}}{\sqrt{4\omega_p \omega_k}} \left\{ i\omega_k f_p^{+*}(t'(0, x)) e^{ip_1 x'_1(0, x)} - \partial_t \left\{ f_p^{+*}(t'(t, x)) e^{ip_1 x'_1(t, x)} \right\} \right\} |_{t=0} \tag{4.4}$$

where

$$\begin{aligned}
x'(t, x) &= \frac{2}{\lambda} \tan^{-1} \left( \frac{\cosh(\frac{\lambda t}{2}) \sin(\frac{\lambda x}{2})}{\cosh(\frac{\lambda t}{2}) \cos(\frac{\lambda x}{2}) \cosh(\frac{\lambda}{2}(T'_0 - T_0)) - \sinh(\frac{\lambda t}{2}) \sinh(\frac{\lambda}{2}(T'_0 - T_0))} \right) \\
t'(t, x) &= \frac{2}{\lambda} \sinh^{-1} \left( \sinh(\frac{\lambda t}{2}) \cosh(\frac{\lambda}{2}(T'_0 - T_0)) - \cosh(\frac{\lambda t}{2}) \cos(\frac{\lambda x}{2}) \sinh(\frac{\lambda}{2}(T'_0 - T_0)) \right). \tag{4.5}
\end{aligned}$$

## V. DISCRETE SHIFT OF ENERGY SPECTRUM

Unfortunately due to the complicated nature of the expression for  $\beta(k, p)$  we cannot write it out in a more transparent form which is still exact. We can however discover some interesting facts about the spectrum of created particles by investigating the integrand of the integral for  $\beta(k, p)$ . In fact it is not difficult to see that the particles observed by our stationary observer possess a discrete energy spectrum shift. To see this we rewrite (4.4) as

$$\beta(k, p) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} dx \frac{e^{i(p_1+q_1)x_1}}{\sqrt{4\omega_p \omega_k}} F(k, p, x) \delta(p_2 + q_2) \delta(p_3 + q_3) \tag{5.1}$$

where



$$F(k, p, x) = e^{ip_1(x'(0, x) - x)} \left\{ i\omega_k f_p^{+*}(t'(0, x)) - iq_1 \frac{\partial x'}{\partial t} f_p^{+*}(t'(t, x)) - \frac{\partial t'}{\partial t} \partial_{t'} (f_p^{+*}(t'(t, x))) \right\} \Big|_{t=0} \quad (5.2)$$

By inspection of the inverse relations (4.5) one sees that  $F(k, p, x)$  is a well behaved periodic function in  $x$ . The only difficulty arises with the exponential factor. This factor is also periodic in  $x$  if one is careful to ensure that in the analysis both  $x'$  and  $x$  retain their range of  $-\infty$  to  $\infty$ .

We can therefore write,

$$F(p, k, x) = \sum_{n=-\infty}^{\infty} C_n(p, k) e^{in \frac{\lambda x}{2}} \quad (5.3)$$

which implies that,

$$\beta(p, k) = \frac{-i}{\sqrt{4\omega_p \omega_k}} \sum_{n=-\infty}^{\infty} C_n(p, k) \delta(p_1 + k_1 + \frac{n\lambda}{2}) \delta(p_2 + k_2) \delta(p_3 + k_3). \quad (5.4)$$

Unfortunately we cannot evaluate the  $C_n(p, k)$  analytically but we can evaluate them numerically for some specific values of  $(T'_0 - T_0), \lambda, \mathbf{p}$  and  $\mathbf{q}$ . This numerical analysis suggests that the particle creation drops off rapidly for large  $\mathbf{p}$  and  $\mathbf{q}$ . Nevertheless, it is expected that the total particle creation, as in all such problems, is infinite. The reason for this seems to be that the external field can pump in an infinite amount of energy in a finite time [4]. In this particular model the energy density of the classical matter field giving rise to the geometry of the model is constant. If one calculates the total energy of the classical matter field it is therefore infinite.

## VI. CONCLUSIONS

We see from the above analysis that the particle creation due to the gravitational field as seen by a stationary observer in the model universe,  $ds^2 = dT^2 - e^{\lambda T} (dX^1)^2 - (dX^2)^2 - (dX^3)^2$ , observes a spectrum of particles shifted by a discrete amount. It appears that the one length scale of the geometry namely  $\sqrt{R}$  plays a role similar to the role the length of a box plays for modes in a box. In this sense the discrete energy spectrum shift may almost be expected.

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